ECE 302: Probabilistic Methods in Electrical and Computer Engineering
Spring 2018
Instructor: Prof. A. R. Reibman

PURDUE UNIVERSITY

Exam 1

Spring 2018, MWF 2:30-3:20pm
(February 12, 2018)

SOLUTIONS

This is a closed book exam. Neither calculators nor help sheets are allowed.

Cheating will result in a zero on the exam and possibly failure of the class. Do not cheat!

Use of any electronics is considered cheating.

Put your name on every page of the exam and turn in everything when time is up.

Name: _____ PUID: _____

I certify that I have neither given nor received unauthorized aid on this exam.

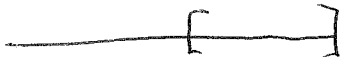
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Problem 1. (MULTIPLE CHOICE: 5 POINTS)
Suppose all outcomes in the sample space $S = \{0, 3\}$ are equally likely. What is the probability that the outcome is between 2 and 4 inclusive?

(a) 0
(b) 1/3
(c) 1/2
(d) 2/3
(e) 3/4
(f) 1
(g) None of the above

S: 

just 2 (discrete) elements

Interval 

→ only 1 of the 2 elements of S are in this interval

Problem 2. (MUTLIPLE CHOICE: 5 POINTS)

Suppose events A and B are independent, and $P(A) = P(B) = 1/2$. What is $P(A \cup B)$?
(If you show your work, you may receive partial credit.)

(a) 0
(b) 1/4
(c) 1/2
(d) 3/4
(e) 1
(f) Impossible to determine.
(g) None of the above

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$ (always)

$= P(A) + P(B) - P(A)P(B)$ (because of independence)

$= \frac{1}{2} + \frac{1}{2} - \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4}$

Problem 3. (TRUE/FALSE: 5 POINTS EACH, TOTAL 15 POINTS)
For each of the following statements, determine which is valid. If you show your reasoning you might get partial credit. Finding a counter-example might be helpful if the answer is FALSE.

Clearly label each statement T or F to the left of the problem number.

T (a) If $B_1, B_2, \ldots, B_n$ are mutually exclusive and collectively exhaustive, then $P(A) = \sum_{i=1}^{n} P(A|B_i)P(B_i)$.

F (b) Let X be a continuous random variable with PDF $f_X(x)$. The probability $P(X = x) = f_X(x)$.

F (c) For any events A and B with $P(A) > 0$ and $P(B) > 0$, $P(A|B) + P(B|A) = 1$.

a) thm total probability
b) $P(X = x) = 0$ for a continuous RV, and $f_X(x)$ could be positive
c) Conditioning on A gives a different sample space than conditioning on B, so there's no relationship.

Problem 4. (MULTIPLE CHOICE: 5 POINTS EACH, TOTAL 10 POINTS)
(If you show your work, you may receive partial credit.)

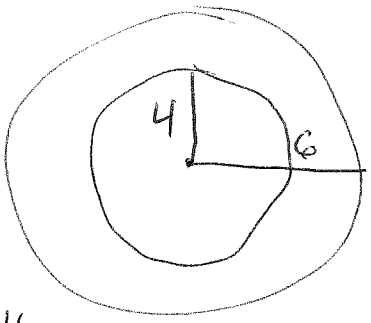
In the Olympic sport of curling, a team works together to slide a "rock" down the ice into a circle on the other side of the ice rink. The goal is to get the rock as close to the center of the circle as possible.
(It's a very strategic game, but we will simplify it here.)

Suppose the first rock slid down the ice has an equally likely chance to stop anywhere in a circle with 6-foot radius about the center.

(a) What is the probability the rock stops within 4 feet of the center?

(a) 1/4
(b) 1/2
(c) 4/9
(d) 2/3
(e) Impossible to determine.
(f) None of the above

$A = \{\text{within 4 ft}\}$
$S = \{\text{within 6 ft}\}$

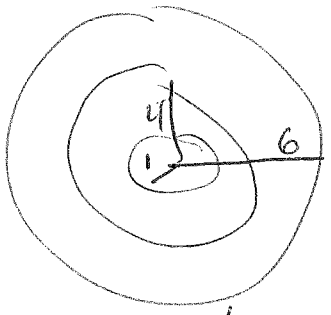


$$P(A) = \frac{|A|}{|S|} = \frac{4^2\pi}{6^2\pi} = \frac{16}{36} = \frac{4}{9}$$

(b) Given that the rock stops within 4 feet of the center, what is the probability the rock stops within 1 foot of the center?

(a) 1/36
(b) 1/4
(c) 1/2
(d) 1/16
(e) Impossible to determine.
(f) None of the above

$B = \{\text{within 1 ft}\}$



2 ways to solve:

① $$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(B)}{P(A)} = \frac{1^2\pi / 6^2\pi}{4^2\pi / 6^2\pi} = \frac{1}{16}$$

② ~~Counting~~ $$= \frac{|B|}{|A|} = \frac{1^2\pi}{4^2\pi} = \frac{1}{16}$$

No yellow Light

Problem 5. (15 POINTS)

Suppose when you drive to the store, you go through two traffic lights that have been synchronized by the traffic engineers. In particular, the timing is designed such that whichever color the driver encounters at the first light, the second light is the same color with probability 3/4.

$P(G_2|R_1) = 3/4$

$P(R_1) = P(G_1) = 1/2$

(a) Assuming the first light is equally likely to be Green or Red, what is the probability $P(G_2)$ that the second traffic light is Green?

$P(R_2|G_1) = 3/4$

(b) What is the probability that you wait for *at least* one light?

(c) What is $P(G_1|R_2)$, the probability the first light was Green given that the second light was Red?

a) $P(G_2) = P(G_2|R_1)P(R_1) + P(G_2|G_1)P(G_1)$

$= (1 - 3/4)(1/2) + (\frac{3}{4})(\frac{1}{2}) = \boxed{\frac{1}{2}}$

b) $P(R_1 \cup R_2) = P(R_1) + P(R_2) - P(R_1 \cap R_2)$

OR $= 1 - P(R_1^c \cap R_2^c)$ (DeMorgan's)

OR $= 1 - P(G_1 \cap G_2)$

$= 1 - P(G_2|G_1)P(G_1) = 1 - (\frac{3}{4})(\frac{1}{2})$

$= 1 - 3/8 = \boxed{5/8}$

c) $P(G_1|R_2) = \frac{P(R_2|G_1)P(G_1)}{P(R_2)} = \frac{P(R_2|G_1)P(G_1)}{P(R_2|G_1)P(G_1) + P(R_2|G_2)P(G_2)}$

(or: $P(R_2) = 1 - P(G_2) = 1/2$ from (a))

so $P(G_1|R_2) = \frac{(1/4)(1/2)}{1/2} = \boxed{\frac{1}{4}}$

Problem 6. (15 POINTS (5 EACH FOR (A,B,C), 2 BONUS POINTS FOR (D))
According to folklore, derived in part from the beliefs of an American Indian tribe, an animal called the groundhog is capable of predicting on February 2 each year whether there will be 6 more weeks of winter. Let W denote the event that there will actually be 6 more weeks of winter.

We will examine step by step 3 groundhogs: Chuck, Filip, and Gertrude. Define C, F, G as the events that Chuck, Filip, and Gertrude each predict 6 more weeks of winter.

Note: If you can't solve part (a) you can leave your result for parts (b,c) in terms of $P(W)$.
Part (d) is a Bonus 2 points.

Hint: the method each groundhog uses to predict doesn't matter **except for Filip!!!**

(a) Filip flips a fair coin and predicts 6 more weeks of winter when the coin is heads. When Filip predicts 6 more weeks of winter (i.e., when he predicts event W), Filip is right 70% of the time.

What is $P(W)$, the probability there will be 6 more weeks of winter?

(b) Gertrude simply scratches at one of two cards placed in front of her; one card predicts 6 more weeks of winter and the other predicts its complement. Gertrude's predictions are never ever right.

When Gertrude predicts 6 more weeks of winter, what is the probability there will be 6 more weeks of winter?

(c) Chuck predicts 6 more weeks of winter when the day is sunny. He predicts 6 more weeks of winter with exactly the same probability that Gertrude does, but Chuck is right 40% of the time.

When Chuck predicts 6 more weeks of winter, what is the probability there will be 6 more weeks of winter?

(d) (Bonus) If every year you were to bet \$100 on whether or not there would be 6 more weeks of winter, which groundhog's prediction would you rather use to decide how to make your bet? (You can use any strategy to decide your bet, but it must rely on your chosen groundhog's prediction.) Describe your strategy.

(Note: space is provided on the next page to solve this.)

a) $P(W|F) = 0.7$

But F and W are independent (via physics)
(a $P(F) = 0.5$ regardless of anything, for a fair coin)

So $P(W|F) = \boxed{P(W) = 0.7}$

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When Chuck predicts 6 more weeks of winter, what is the probability there will be 6 more weeks of winter?

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(Note: space is provided on the next page to solve this.)

b) Gertrude is never right $\Rightarrow \boxed{P(W|G) = 0}$

This is the answer to b), but for c) we need $P(G)$.

Create a probability table. $P(W^c|G^c) = 0$ also

	W	W^c	
G	0		
G^c		0	
	0.7	0.3	

$P(G)$ $P(G) = P(G \cap W) + P(G \cap W^c)$

$= 0 + [P(W^c) - P(G^c \cap W^c)]$

$= 0.3 - 0 = 0.3 = P(G) = 1 - P(W)$

(for part c)

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According to folklore, derived in part from the beliefs of an American Indian tribe, an animal called the groundhog is capable of predicting on February 2 each year whether there will be 6 more weeks of winter. Let W denote the event that there will actually be 6 more weeks of winter.

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Note: If you can't solve part (a) you can leave your result for parts (b,c) in terms of $P(W)$.
Part (d) is a Bonus 2 points.

Hint: the method each groundhog uses to predict doesn't matter **except for Filip!!!**

(a) Filip flips a fair coin and predicts 6 more weeks of winter when the coin is heads. When Filip predicts 6 more weeks of winter (i.e., when he predicts event W), Filip is right 70% of the time.

What is $P(W)$, the probability there will be 6 more weeks of winter? $P(W) = 0.7$

(b) Gertrude simply scratches at one of two cards placed in front of her; one card predicts 6 more weeks of winter and the other predicts its complement. Gertrude's predictions are never ever right.

When Gertrude predicts 6 more weeks of winter, what is the probability there will be 6 more weeks of winter? $P(W|G) = 0$ and $P(G) = 0.3 = 1 - P(W)$

(c) Chuck predicts 6 more weeks of winter when the day is sunny. He predicts 6 more weeks of winter with exactly the same probability that Gertrude does, but Chuck is right 40% of the time.

When Chuck predicts 6 more weeks of winter, what is the probability there will be 6 more weeks of winter?

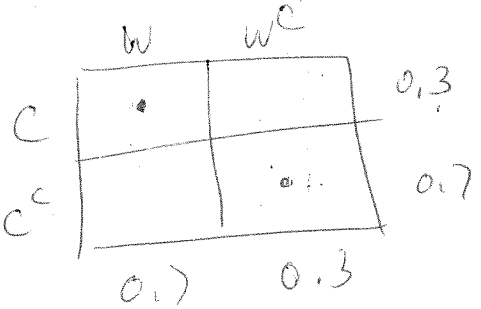
(d) (Bonus) If every year you were to bet \$100 on whether or not there would be 6 more weeks of winter, which groundhog's prediction would you rather use to decide how to make your bet? (You can use any strategy to decide your bet, but it must rely on your chosen groundhog's prediction.) Describe your strategy.

(Note: space is provided on the next page to solve this.)

c)

$P(C) = P(G) = 0.3$

$P(C \cap W) + P(C^c \cap W^c) = 0.4$
(because right 40% of time)



$P(W|C) = \dfrac{P(W \cap C)}{P(C)}$

but $P(W \cap C) + P(W^c \cap C^c) = 0.4$

$P(W) = P(W \cap C) + P(W \cap C^c) = 0.7$

(sum) $2P(W \cap C) + P(C^c) = 1.1$

$2P(W \cap C) = 1.1 - 0.7 = 0.4$

$P(W \cap C) = 0.2$

so $P(W|C) = \dfrac{P(W \cap C)}{P(C)} = \dfrac{0.2}{0.3}$

$= \boxed{2/3}$

Problem 7. (15 POINTS)

A well-shuffled deck contains 52 cards, 4 of which are Queens.
Suppose 7 cards are deal to each of 3 people; each pile of 7 cards is called a "hand".

Note: you may leave your answer in terms of factorials, or even $\binom{n}{k}$ notation.

(a) Let H be the set that contains all possible hands. How many elements are in the set H? That is, how many ways can 7 cards be dealt from a deck of 52 cards?

(b) What is the probability that any one given hand contains no Queens.

(c) What is the probability that none of the 3 hands contains a Queen.

a) $|H| = \binom{52}{7} = \frac{52!}{45!\,7!} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot 47 \cdot 46}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}$

b) One hand has 7 cards.
So the sample space is H, since there's no replacement when you deal cards.
There should be no queens, so the size of the event of interest is $\binom{48}{7}$, since there are 48 cards that aren't queens.

$$P(\text{no Q one hand}) = \binom{48}{7} \Big/ \binom{52}{7} = \frac{48!}{41!\,7!} \cdot \frac{45!\,7!}{52!}$$

$$= \frac{48!}{51!} \cdot \frac{45!}{41!}$$

c) Same like part (b), except you deal $3 \cdot 7 = 21$ cards without Queens.
Denominator: $\binom{52}{21}$ Numerator $\binom{48}{21}$

Problem 8. (20 POINTS)

There are 4 parts to this problem. Part (d) is on the next page!!

An Olympic athlete competes in the biathalon, a sport which combines both cross country skiing and shooting.

The time it takes her to complete a particular ski course is a random variable that depends on the amount of sun and clouds, the temperature, and any competitors who may block her path. Let X be a random variable that describes the time it takes her to complete the course, in minutes.

In addition, the number of targets she hits during the shooting is a random variable, denoted Y.

(a) Describe the two random variables. (Multiple choice. Mark your answer clearly.)

(i) X takes on discrete values and Y takes on discrete values.

(ii) X takes on continuous values and Y takes on discrete values.

(iii) X takes on discrete values and Y takes on continuous values.

(iv) X takes on continuous values and Y takes on continuous values.

(v) None of the above.

a) minutes are a unit. ~~Time~~ Underlying physical process is continuous time. # targets is discrete

(b) Suppose the PDF of X is given by

$$f_X(x) = \begin{cases} c(x-40) & \text{for } 40 \le x \le 60 \\ 0 & \text{otherwise} \end{cases}$$

Find the value of the constant c.

(c) For the same PDF as in part (b), compute the probability she finishes in between 45 and 55 minutes.

b) $\int_{-\infty}^{\infty} f_X(x)\,dx = 1 \Rightarrow \int_{40}^{60} c(x-40)\,dx = \int_0^{20} cy\,dy = \frac{cy^2}{2}\Big|_0^{20}$

$= \frac{c(20)^2}{2} = 1 \Rightarrow c = \boxed{\frac{1}{200}}$

c) $\int_{45}^{55} c(x-40)\,dx = \int_5^{15} cy\,dy = \frac{cy^2}{2}\Big|_5^{15} = \frac{1}{400}\left(15^2 - 5^2\right)$

$= \frac{1}{400}(225 - 25) = \frac{200}{400} = \boxed{1/2}$

Don't forget part (d) on the next page!

Review of CDF

(d) Suppose the CDF of Y, the number of targets she hits, is given by

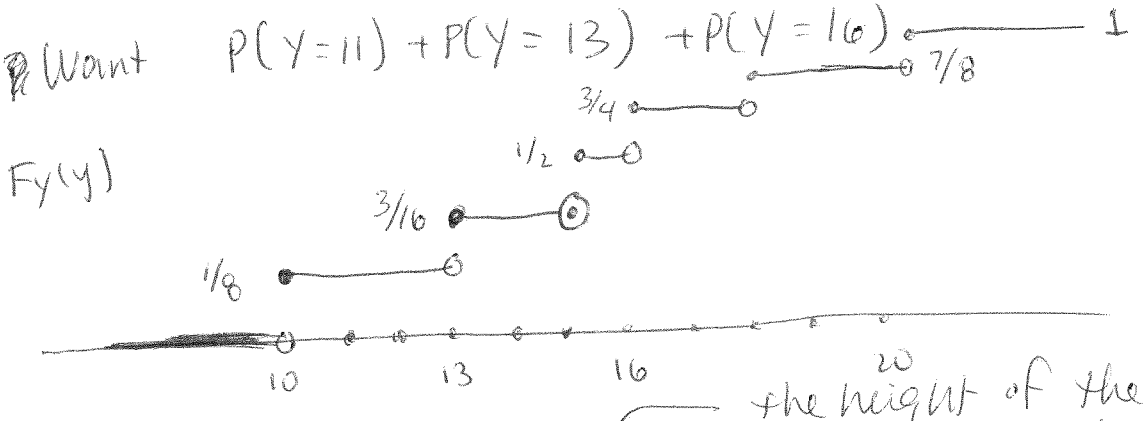
$$F_Y(y) = \begin{cases} 0 & \text{for } y < 10 \\ 1/8 & \text{for } 10 \le x < 13 \\ 3/16 & \text{for } 13 \le x < 15 \\ 1/2 & \text{for } 15 \le x < 16 \\ 3/4 & \text{for } 16 \le x < 18 \\ 7/8 & \text{for } 18 \le x < 20 \\ 1 & x \ge 20 \end{cases}$$

all y's

What is the probability she hits either 11, 13, or 16 targets?

$F_Y(y) = P(Y \le y)$

Want $P(Y=11) + P(Y=13) + P(Y=16)$



$P(Y=y) = P(Y=y) - P(Y=y^-)$, the height of the jump

or for discrete with integer sample space, $P(Y=y) - P(Y=y-1)$

$P(Y=13) = F_Y(13) - F_Y(12) = 3/16 - \frac{1}{8} = \frac{1}{16}$

$P(Y=11) = F_Y(11) - F_Y(10) = \frac{1}{8} - \frac{1}{8} = 0$

$P(Y=16) = F_Y(16) - F_Y(15) = \frac{3}{4} - \frac{1}{2} = \frac{1}{4}$

$P(Y=11) + P(Y=13) + P(Y=16) = 0 + 16 + \frac{1}{4} = \boxed{\frac{5}{16}}$