ECE 302: Probabilistic Methods in Electrical and Computer Engineering
Fall 2017
Instructor: Prof. A. R. Reibman

PURDUE UNIVERSITY

Exam 1

Fall 2017, TTh 3-4:15pm
(October 5, 2017)

SOLUTIONS

This is a closed book exam. Neither calculators nor help sheets are allowed.

Cheating will result in a zero on the exam and possibly failure of the class. Do not cheat!

Use of any electronics is considered cheating.

Put your name on every page of the exam and turn in everything when time is up.

Name: _____ PUID: _____

I certify that I have neither given nor received unauthorized aid on this exam.

Signature: _____

Problem 1. (15 POINTS) (A IS 10 POINTS; B IS 5 POINTS)
Among the workstations available to a group of students in a computing course, let C be the set of workstations with 4 or more cores, and M be the set of workstations with 16 GBytes or more of memory.
40% of workstations have four or more cores, while 60% have 16 GBytes or more of memory. Of those workstations with more than 16 GBytes or more of memory, five eighths (5/8-ths) of them also have 4 or more cores.

↳ conditional

(a) What is the probability a randomly chosen workstation is in both sets C and M, or is in neither?

(b) Are the events C and M independent? Prove your answer with adequate justification.

$C = \{$ workstation has 4 or more cores $\}$
$M = \{$ workstation has 16 GBytes or more memory $\}$

$P(M) = 0.6$
$P(C) = 0.4$
$P(C|M) = 5/8$

a) find $P((M \cap C) \cup (M^c \cap C^c))$
$= P(M \cap C) + P(M^c \cap C^c)$ because they're disjoint
$= P(M \cap C) + 1 - P(M \cup C)$
$= P(M \cap C) + 1 - [P(M) + P(C) - P(M \cap C)]$
$= P(M \cap C) + 1 - P(M) - P(C) + 2P(M \cap C)$
$= 1 - P(M) - P(C) + 2P(C|M)P(M) = 2\left(\frac{5}{8}\right)\left(\frac{6}{10}\right)$
$= 1 - 0.6 - 0.4 + 2P(C|M)P(M)$
$= \boxed{3/4}$

b) If independent, then
$P(M \cap C) = P(M)P(C)$
But $P(M \cap C) = P(C|M)P(M) = \left(\frac{5}{8}\right)\left(\frac{6}{10}\right) = \frac{3}{8}$
and $P(M)P(C) = \left(\frac{6}{10}\right)\left(\frac{4}{10}\right) = \frac{24}{100} = \frac{6}{25}$

These are not equal, therefore M and C are not independent.

Problem 2. (15 POINTS (A IS 8 POINTS, B IS 7 POINTS))
You flip a unfair, weighted coin, that has $P(head) = 1/4$. If the result is a heads, you then roll a fair 4-sided die. If the result is a tails, you then roll a fair 6-sided die.

(a) What is the probability your die-roll results in a 2?

(b) Given that the die-roll results in a 2, what is the probability you first got a tail?

A sequential experiment.

$P(H) = 1/4, \quad P(T) = 3/4.$

let $R_2 = \{\text{roll a 2}\}$. $\quad P(R_2|H) = \frac{1}{4}$. $\quad P(R_2|T) = \frac{1}{6}$.

a) $P(R_2) = P(R_2|H)P(H) + P(R_2|T)P(T)$ by the theorem of total probability

$= \frac{1}{4}\,\frac{1}{4} + \frac{1}{6}\,\frac{3}{4}$

$= \frac{1}{16} + \frac{3}{24} = \boxed{\frac{3}{16}}$

b) $P(T|R_2) = \dfrac{P(R_2|T)\,P(T)}{P(R_2)}$ by Bayes Rule

$= \dfrac{\frac{1}{6}\,\frac{3}{4}}{\frac{3}{16}} = \frac{3}{24}\,\frac{16}{3} = \boxed{\frac{2}{3}}$

Problem 3. (5 POINTS (MULTIPLE CHOICE))
A 5-bit codeword is sent over a noisy channel, where bits are flipped with probability p, each independently. An error correcting code is designed such that if two or fewer bits of a codeword are flipped, the codeword can be correctly decode.

What's the probability a codeword cannot be decoded?
(If you show your work, you may receive partial credit.)

(a) $\binom{5}{2}p^2(1-p)^3$

(b) $p^5(1-p)$

(c) $\binom{5}{3}p^3(1-p)^2 + 5p^4(1-p) + p^5$

(d) $p^2(1-p)^3$

(e) $\binom{5}{3}p^3(1-p)^2$

(f) None of the above

X = # flipped bits out of 5 bits.

Binomial $n = 5$, $p = p$.

so $P_X(k) = \binom{5}{k} p^k (1-p)^{5-k}$

BUT we want

$P(X \geq 3) = 1 - P(X \leq 2)$

$= P(X=3) + P(X=4) + P(X=5)$

$= P_X(3) + P_X(4) + P_X(5)$

$= \binom{5}{3} p^3 (1-p)^2 + \binom{5}{4} p^4 (1-p) + \binom{5}{5} p^5$

$\Rightarrow$ answer c

Problem 4. (15 POINTS (A IS 8 POINTS AND AND B IS 7 POINTS))
Pallavi, Varsha, and Smita repair amplifiers at a local music shop. Pallavi is still in training, so she repairs 20% of the amplifiers and has a 60% success rate (meaning that 60% of her repairs are effective). Varsha and Smita, experienced pros, each repair 40% of the amplifiers and each have a 90% success rate.

(b) What is the probability that an amplifier does not get properly repaired?

(a) Given that an amplifier was defective after being repaired, what is the probability that Pallavi did the repair?

Let $N = \{$ Novice does repair $\}$ $\quad P(N) = \frac{2}{10}$

$P(N^c) = \frac{8}{10}$

Let $F = \{$ amplifier is fixed $\}$

$P(F|N) = 6/10$

$P(F|N^c) = 9/10$

The rem of total probability

a) $P(F^c) = P(\{$ amplifier is NOT fixed $\})$

$= P(F^c|N)\,P(N) + P(F^c|N^c)\,P(N^c)$

$= [1 - P(F|N)]\,P(N) + [1 - P(F|N^c)]\,P(N^c)$

$= \frac{4}{10}\frac{2}{10} + \frac{1}{10}\frac{8}{10} = \frac{8}{100} + \frac{8}{100} = \boxed{\frac{16}{100}}$

b) $P(N|F^c) = \dfrac{P(F^c|N)\,P(N)}{P(F^c)}$

$= \dfrac{\frac{4}{10}\frac{2}{10}}{\frac{16}{100}} = \boxed{\frac{1}{2}}$

Problem 5. (20 POINTS)
A discrete random variable X has the following probability mass function (PMF):

$$p_X(x) = \begin{cases} cx & \text{for } x = 1, 2, 3, 4. \\ 1/2 & \text{for } x = 5 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the value of the constant c.

(b) Find $E(X)$.

(c) Find $Var(X)$.

(d) What is the largest value of x for which $P(X > x) > 0.75$? (Hint, a sketch of the PMF can be helpful.)

a) $\sum_{\text{all } x} p_X(x) = 1$ so $1 = c\cdot[1+2+3+4] + \frac{1}{2}$

$1 = 10c + 1/2$

$\boxed{c = 1/20}$

b) $E(X) = \sum_{\text{all } x} x\, p_X(x)$

$= c[1\cdot 1 + 2\cdot 2 + 3\cdot 3 + 4\cdot 4] + \frac{1}{2}\cdot 5$

$= c(30) + \frac{5}{2} = \frac{30}{20} + \frac{5}{2} = \boxed{4}$

c) $Var(X) = E(X^2) - E(X)^2$

$E(X^2) = \sum_{\text{all } x} x^2 p_X(x) = c[1\cdot 1^2 + 2\cdot 2^2 + 3\cdot 3^2 + 4\cdot 4^2] + \frac{5^2}{2}$

$= c(1+8+27+64) + \frac{25}{2}$

$= c(100) + \frac{25}{2} = 5 + 12.5 = \frac{35}{2}$

so $Var(X) = \frac{35}{2} - (4)^2 = \frac{35}{2} - \frac{32}{2} = \boxed{\frac{3}{2}}$

d) on next page

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(a-c) on previous page

d) $P(X > x) = \sum_{k>x} P(X = k) = \sum_{k>x} P_X(k)$

make a table

k	$P_X(k)$	$P(X > x)$	
0	0	$1/20 + 2/20 + 3/20 + 4/20 + 1/2 = 1$	
1	$1/20$	$2/20 + 3/20 + 4/20 + 1/2 = 19/20$	$= 0.95$
2	$2/20$	$3/20 + 4/20 + 1/2 = 17/20$	$= 0.85$
3	$3/20$	$4/20 + 1/2 = 14/20$	$= 0.70$
4	$4/20$	$1/2$	$= 0.5$
5	$1/2$	0	0

So $P(X > x)$ is larger than 0.75 for $x = 0$, $x = 1$, $x = 2$, and the largest of these is $\boxed{x = 2}$

Problem 6. (10 POINTS)
Let X be a random variable with mean μ and variance σ^2. Define two new random variables, $Y = 2 - X$ and $Z = 3X + 1$. Express $Var(X + Y + Z)$ in terms of μ and σ^2.

$Y = 2 - X$

$Z = 3X + 1$

$X + Y + Z = X + (2 - X) + (3X + 1) = 3X + 3$

$Var(3X + 3) = Var(3X) + Var(3)$

$= 9\,Var(X) + 0$

$= \boxed{9\sigma^2}$ by applying properties of variance.

Note: $Var(X + Y + Z) \neq Var(X) + Var(Y) + Var(Z)$ in general!

Problem 7. (5 POINTS, MULTIPLE CHOICE)
Let $A = [0, 3]$, $B = [-1, 1]$, and $C = (2, 4)$ be sets of real numbers. Simplify $A \cap (B \cup C)$. (If you show your work, you may receive partial credit.)

(a) $[0, 1] \cup [2, 3]$

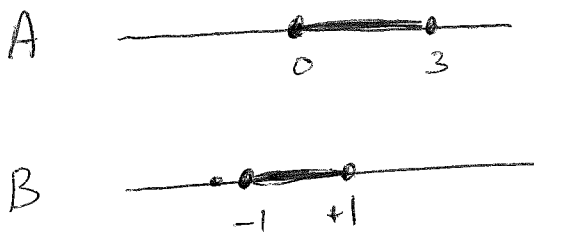
(b) $[0, 3]$

(c) $[0, 1] \cup (2, 3]$

(d) $[-1, 4)$

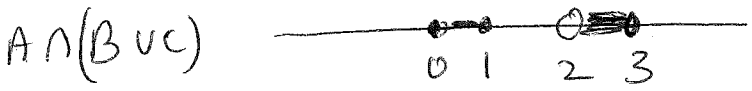
(e) $[-1, 0] \cup (3, 4)$

(f) None of the above









$[0,1] \cup (2,3]$

So answer c

Problem 8. (TRUE/FALSE: 5 POINTS EACH, TOTAL 15 POINTS)
For each of the following statements, determine which is valid for events A, B, and C. If you show your reasoning you might get partial credit. Use a Venn diagram only if it is helpful. Finding a counter-example might be helpful if the answer is FALSE.

Clearly label each statement T or F to the left of the problem number.

T (a) $P(A^c \cup B) \leq 1 - P(A) + P(B)$ for any A and any B.

F (b) If A and B are independent, and B and C are independent, then A and C are also independent.

F (c) If A, B, and C are mutually exclusive and collectively exhaustive, then A^c, B^c, and C^c are also mutually exclusive and collectively exhaustive.

a) $P(A^c \cup B) = P(A^c) + P(B) - P(A^c \cap B)$

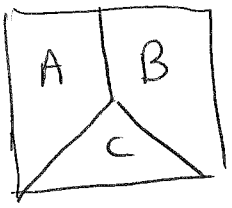
$= 1 - P(A) + P(B) - P(A^c \cap B)$

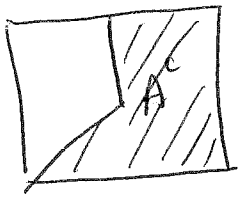
this term is greater than or equal to zero, so if we remove it, the RHS (right hand side) gets bigger. So $P(A^c \cup B) \leq 1 - P(A) + P(B)$

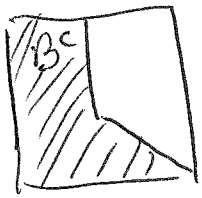
TRUE

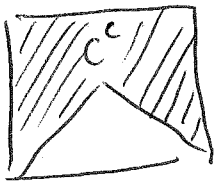
b) False A simple counter example: Let $A = C$. A and C are not independent.

c) False









These are not disjoint (mutually exclusive) although they are collectively exhaustive