ECE 302: Probabilistic Methods in Electrical and Computer Engineering
Fall 2017
Instructor: Prof. A. R. Reibman

PURDUE UNIVERSITY

Sample Questions for Exam 1

Fall ~~2017~~, TTh 3-4:15pm
(October 2017)

SOLUTIONS

These are a collection of problems that have appeared in either Prof. Reibman's real midterm exams or "sample exams." These can all be solved by applying the material we covered in class so far this semester, and together cover all the concepts to be tested on the October exam. Note that this does not mean the problems on the exam will be the same, just that topic coverage is similar. The point values are leftover from the original exam.
The last page of this document will be provided to you as the last page of the exam. This will be all the formulas that will be avaiable to you. The rest you must memorize.

Note that some problems will appear familiar; you've seen them for Homework.

Exam coverage:
Sections 2.1-2, 2.4-6. Sections 3.1-3.3, 3.5.

#1

Problem 1. (10 POINTS)

Among the Purdue students taking ECE 302 this semester, some like dogs, some like cats, some like both, and some like neither. Let D be the set of Purdue ECE302 students who like dogs, and C be the set who like cats.

A study shows that 22% like both cats and dogs, and 12% like neither. The probability a student likes dogs exceeds the probability a student likes cats by 0.14. What is the probability a randomly chosen student likes cats?

$D = \{$ likes dogs $\}$

$C = \{$ likes cats $\}$

	C	C^c	
D	0.22		$x + 0.14$
D^c		0.12	$1 - x - 0.14$
	x	$1-x$	

x ← value to find

$P(C \cap D) = 0.22$

$P(C^c \cap D^c) = 0.12$

$P(C \cup D) = 1 - P(C^c \cap D^c)$

$= 1 - 0.12$

$= 0.88$

$= P(C) + P(D) - P(C \cap D)$

$= x + (x + 0.14) - 0.22$

$0.88 + 0.22 = 2x + 0.14$

$1.10 - 0.14 = 2x$

$\frac{1}{2}(.96) = x = \boxed{0.48}$

Can complete the table now

	C	C^c	
D	0.22	0.40	0.62
D^c	0.26	0.12	0.38
	0.48	0.52	

#2

Problem 4. (10 POINTS)
Among the students in a class, 60% prefer dark chocolate to milk chocolate, 70% prefer basketball to football, and 40% prefer both dark chocolate and basketball.
Determine the probability that a randomly selected student prefers milk chocolate and football.

starting point →

	D	D^c	
B	.4		.7
B^c			
	.60		

$D = \{\text{prefer dark}\}$
$D^c = \{\text{prefer milk}\}$

$B = \{\text{prefer basket ball}\}$
$B^c = \{\text{prefer football}\}$

$P(D \cap B^c) = 0.6 - 0.4 = 0.2$
$P(D^c \cap B) = 0.7 - 0.4 = 0.3$

	D	D^c
B	.4	.3
B^c	.2	

All 4 probabilities must sum to 1
⇒ remaining
$P(B^c \cap D^c) = 0.1$

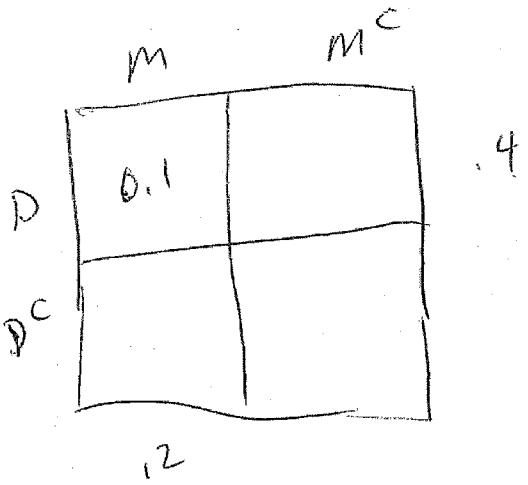
#3

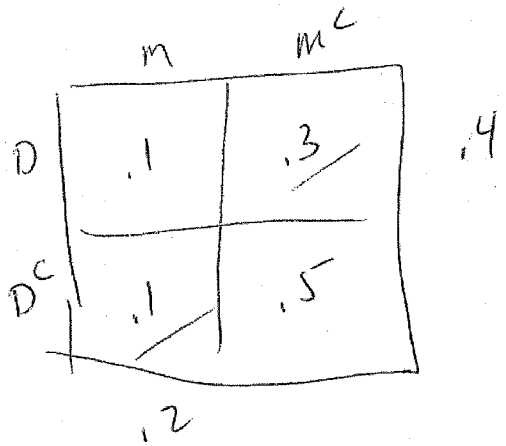
Problem 3. (7 POINTS)
Computers purchased from the surplus store will experience hard drive failures with probablitiy 0.4, memory failures with probability 0.2, and experience both types at the same time with probability 0.1.

What is the probability that there will be either a hard drive failure or a memory failure, but not both.

D = {disk fail}
M = {memory fail}

Given





$$P((D \cap M^c) \cup (D^c \cap M)) = 0.1 + 0.3 = 0.4$$

#4

Problem 4. (5 POINTS EACH, 20 POINTS TOTAL)
Consider the x-y plane. Consider a disk (circle) of radius 2 on the plane, which are those points satisfying $x^2 + y^2 \leq 2$. We throw a dart at the disk and we know that the dart will land uniformly likely on the disk. Let X and Y denote the x and y coordinates of the landing location of the dart. Answer the following questions:

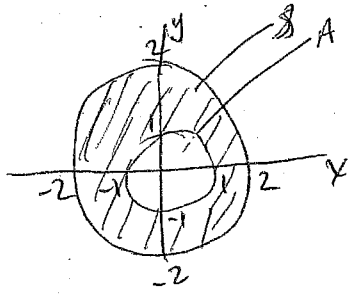
(a) Consider an event $A = \{X^2 + Y^2 \geq 1\}$. What is the probability of the event A?

(b) Consider an event $B = \{X + Y > 0\}$. What is the probability of the event B?

(c) Are events A and B independent?

(d) Consider an event $C = \{X > 1\}$. Are events A and C independent?

a)



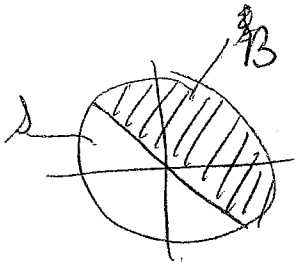
Area($\mathcal{S}$) = 4π

Area(A) = Area($\mathcal{S}$) - Area(inner circle)

$= 4\pi - \pi = 3\pi$

$P(A) = 3\pi / 4\pi = 3/4$

b)



$P(B) = \frac{\text{Area}(B)}{\text{Area}(\mathcal{S})} = 1/2$

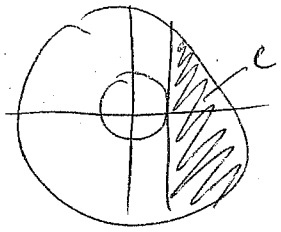
c) Independence requires $P(A \cap B) = P(A)P(B)$ and $P(A) \neq 0$, $P(B) \neq 0$

The latter two are true.

$P(A \cap B) = \frac{\text{Area}(A \cap B)}{4\pi} = \frac{3\pi/2}{4\pi} = 3/8$

$P(A)P(B) = (3/4)(1/2) = 3/8$ so yes they are independent

d)



$P(A \cap C) = P(C) \neq P(A)P(C)$

$\Rightarrow$ No, they are not independent

#5

~~Problem 2.~~ (15 POINTS)

Let A and B be events and let $P(A) = 1/4$ and $P(B) = 1/3$.

(a) Compute $P(A \cup B)$ assuming $A \cap B = \phi$.

(b) Compute $P(A \cup B)$ assuming A and B are independent.

(c) Compute $P(A \cup B)$ assuming $P(B|A) = 2/3$.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

a) $P(A \cap B) = P(\phi) = 0$

so $P(A \cup B) = \frac{1}{4} + \frac{1}{3} + 0 = \boxed{\frac{7}{12}}$

b) $P(A \cap B) = P(A)\,P(B) = \frac{1}{4} \cdot \frac{1}{3} = \frac{1}{12}$

so $P(A \cup B) = \frac{1}{4} + \frac{1}{3} - \frac{1}{12} = \frac{6}{12} = \boxed{\frac{1}{2}}$

c) $P(A \cap B) = P(B|A)\,P(A) = \frac{2}{3} \cdot \frac{1}{4} = \frac{2}{12}$

so $P(A \cup B) = \frac{1}{4} + \frac{1}{3} - \frac{2}{12} = \boxed{\frac{5}{12}}$

#6

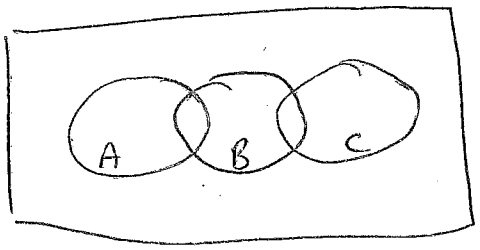
~~Problem 2.~~ (20 POINTS (A AND B ARE EACH 5 POINTS; C IS 10 POINTS))
Let A, B, and C be events in a sample space, where $P(A) = 0.2$, $P(B) = 0.1$, and $P(C) = 0.3$ In addition, A and B are pairwise independent, B and C are pairwise independent, and A and C are disjoint.

(a) Draw a Venn diagram indicating a possible configuration of these three events.

(b) Find $P(A \cap B)$.

(c) Find $P(A \cup B \cup C)$

a)



A and B must intersect, because independent and $P(A) \neq 0$, $P(B) \neq 0$

B and C must intersect for similar reasons,

But A and C do not intersect

b) $P(A \cap B) = P(A)P(B)$ because A and B are independent

$= (0.2)(0.1)$

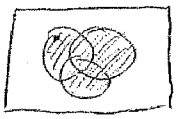
$= \boxed{0.02}$

c) $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

$= 0.2 + 0.1 + 0.3 - 0.02 - 0 - (0.1)(0.3) - 0$

$= \boxed{0.55}$

Problem 7


$A \cup B \cup C$

~~Problem 1.~~ (MULTIPLE CHOICE: 5 POINTS)
Consider three events, A, B, and C, with sample space S. Which of the following are **always** correct ways to compute $P(A \cup B \cup C)$? Hint: Draw a Venn diagram to help you visualize this.

(a) $P(A) + P(A^c \cap B) + P(A^c \cap B^c \cap C)$

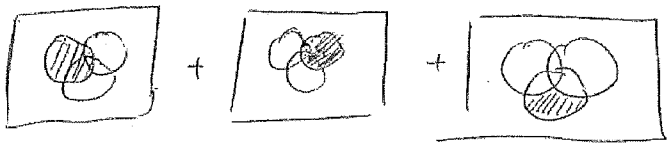
(b) $1 - P(A^c \cap B^c \cap C^c)$

(c) $P(A) + P(B) + P(C)$

(d) Both (b) and (a).

(e) None of the above.

(f) All of the above.

(a) 

Note for (c), if $A \cap B \neq \phi$ ~~and~~ or $A \cap C \neq \phi$ or $B \cap C \neq \phi$ (c) is not true.

Problem 2. (MULTIPLE CHOICE: 5 POINTS)
Which of the following are valid event spaces, for a sample space S. A and B are events.

(a) ϕ

(b) $\{\phi, S\}$

(c) $\{\phi, A, A^c, S\}$

(d) $\{\phi, A, B, S\}$

(e) (b) and (c)

(f) None of the above

A valid event space contains ϕ, S, and if it contains A it also contains A^c and if it contains A and B it also contains $A \cup B$.

However, a valid event space may not contain all events in S. So (b) and (c) are both valid and (d) is correct.

Problem #8

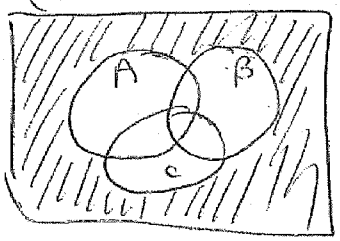
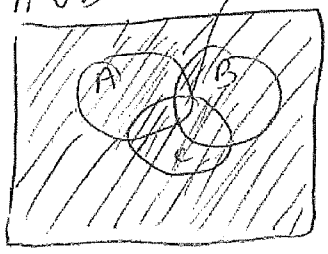
~~Problem 3.~~ (TRUE/FALSE: 5 POINTS EACH, TOTAL 20 POINTS)
For each of the following relations, determine which is valid for arbitrary events A, B, and C. (Note: to be true "for arbitrary events", it must be true for any such event. Use a Venn diagram if it is helpful.) Label each statement T or F to the left of the problem number.

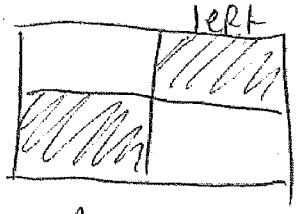
F (a) $(A \cup B \cup C)^c = A^c \cup B^c \cup C^c$

T (b) $(A \cup B) \cap (A^c \cup B^c) = (A \cap B^c) \cup (A^c \cap B) + (A^c \cap B \cap C^c)$

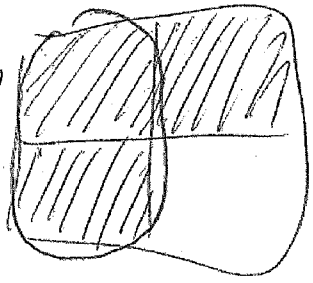
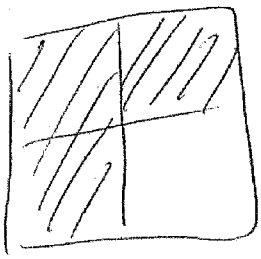
T (c) $(A \cap B) \cup (A \cap B^c) \cup (A^c \cap B) = (A^c \cap B^c)^c$

F (d) $(A - B) - C = A - (B - C)$.

a) $(A \cup B \cup C)^c$  vs $A^c \cup B^c \cup C^c$  all except this little piece

b)  left

Note: + often means union and $(A^c \cap B \cap C^c) \subset (A^c \cap B)$ so this piece is already included. $\Rightarrow$ T

c) A, B  vs 

d) example: Let $B = C$,
then $(A - B) - C = A - B$
but $A - (B - C) = A - \phi = A$

#9

Problem 1. (TRUE/FALSE: 5 POINTS EACH, TOTAL 30 POINTS)
(Note: If a statement is not true for all events, then it is FALSE. Use a Venn diagram if it is helpful.) Label each statement T or F to the left of the problem number.

F (a) If two events A and B are independent, then $P(A|B) = P(A)/P(B)$. X $P(A|B) = P(A)$

T (b) If two events A and B are disjoint, then $P(A \cup B) = P(A) + P(B)$

F (c) If two events A and B are disjoint, they must also be independent. only if $P(A)=0$ or $P(B)=0$

F (d) If two events A and B are collectively exhaustive then $P(A) + P(B) = 1$. Not if $P(A \cap B) \neq \phi$

T (e) If two events A and B are collectively exhaustive then $P(A \cup B) = 1$.

T (f) If two evens A and B are independent, then A^c and B^c are also independent.

(f) Know $P(A \cap B) = P(A)P(B)$

Want $P(A^c \cap B^c) = (1-P(A))(1-P(B))$

$$P(A^c \cap B^c) = P\left((A \cup B)^c\right)$$

$$= 1 - P(A \cup B)$$

$$= 1 - \left(P(A) + P(B) - P(A \cap B)\right)$$

$$= 1 - P(A) - P(B) + P(A)P(B)$$

$$= \left(1 - P(A)\right) - P(B)\left(1 - P(A)\right)$$

$$= \left(1 - P(A)\right)\left(1 - P(B)\right)$$

✓

#10

Problem 6. (5 POINTS EACH, TOTAL 10 POINTS)
Throw a fair dice and toss a fair coin. Let X and Y denote the outcomes of the dice and the coin respectively, where we use the convention that Y = 1 if the outcome of the coin is head. Y = 0 if the outcome of the coin is tail.

(a) What is the sample space in this experiment?

(b) What is the probability that $X^2 + Y$ is a prime number? (Note that 1 is NOT a prime number. The smallest prime number is 2.)

a) sample space $\{(1,0), (1,1), (2,0), (2,1), (3,0), (3,1), (4,0), (4,1), (5,0), (5,1), (6,0), (6,1)\}$

b) $\{x^2+y$ prime $\} = \{1, 2, 4, 5, 9, 10, 16, 17, 25, 26, 36, 37\}$

Assuming a fair coin and fair dice, all outcomes in sample space have equal probability 1/12.

There are 4 outcomes with x^2+y = prime

$P(x^2+y = \text{prime}) = 4/12 = 1/3$

#11

~~Problem 4.~~ (20 POINTS (A AND B ARE 7 POINTS; C IS 6 POINTS))
In a given binary communcation channel, it is equally likely to send either a 1 or a 0. The probability of error given that a 1 was sent is 2/9, while the probability of an error given that a 0 was sent is 1/9.

(a) What is the probability that a 1 is received?

(b) What is the probability that an error occurs?

(c) What is the probability that a 1 was sent, given that a 1 was received?

$S_1 = \{\text{send 1}\}$ $S_0 = \{\text{send 0}\}$ $P(S_1) = P(S_0) = 1/2$

$R_1 = \{\text{receive 1}\}$ $R_1 = \{\text{receive 1}\}$ $P(R_1 | S_0) = 1/9$

$P(R_0 | S_1) = 2/9$

a) $P(R_1) = P(R_1|S_1)P(S_1) + P(R_1|S_0)P(S_0)$

$= \left(1 - \frac{2}{9}\right)\frac{1}{2} + \left(\frac{1}{9}\right)\frac{1}{2} = \frac{7+1}{18} = \boxed{\frac{4}{9}}$

b) $P(\text{error}) = P(R_1 \cap S_0) + P(R_0 \cap S_1)$

$= P(R_1|S_0)P(S_0) + P(R_0|S_1)P(S_1)$

$= \frac{1}{9}\frac{1}{2} + \frac{2}{9}\frac{1}{2} = \frac{3}{18} = \boxed{\frac{1}{6}}$

c) $P(S_1|R_1) = \dfrac{P(R_1|S_1)P(S_1)}{P(R_1)}$ ← use answer from part (a)

$= \dfrac{\left(1 - \frac{2}{9}\right)\frac{1}{2}}{4/9} = \dfrac{7/18}{4/9} = \boxed{\frac{7}{8}}$

Another approach: fill out the probability table, using, for example, $P(R_1 \cap S_0) = P(R_1|S_0)P(S_0)$

	S_0	S_1	
R_0	8/18	2/18	
R_1	1/18	7/18	
	1/2	1/2	

#12

~~Problem~~ 6. (20 POINTS)

You have 4 otherwise identical cans of soda (also known as pop or cola), except you know that 1 was shaken up about 10 minutes ago, while the other 3 have been stable for hours. (You have lost track of which can is which.) The probability of the shaken can splattering when opened is 4/5, and the probability of a stable can splattering when opened is 1/3.

(a) If you choose one can at random and open it, what is the probability of it splattering?

(b) If you open a can and it splatters, what is the probability that it was the shaken can?

(You may leave your answers in fractional form.)

$(\text{shaken})^c = \text{stable}$

Know: $P(\text{splat} \mid \text{shaken}) = 4/5$

$P(\text{splat} \mid \text{stable}) = 1/3$

$P(\text{shaken}) = 1/4$

a) $P(\text{splat}) = P(\text{splat} \mid \text{shaken})\,P(\text{shaken}) + P(\text{splat} \mid \text{stable})\,P(\text{stable})$

$= \frac{4}{5} \cdot \frac{1}{4} + \frac{1}{3}\left(1 - \frac{1}{4}\right)$

$= \frac{1}{5} + \frac{1}{4} = 9/20$

b) $P(\text{shaken} \mid \text{splat}) = \dfrac{P(\text{splat} \mid \text{shaken})\,P(\text{shaken})}{P(\text{splat})}$

$= \dfrac{\left(\frac{4}{5} \cdot \frac{1}{4}\right)}{9/20} = \dfrac{1/5}{9/20} = \frac{4}{9}$

#13

~~Problem 5~~. (8 POINTS EACH, TOTAL 40 POINTS)
Roll 2 fair 6-sided dice.

(a) What is the probability that you roll doubles (both die show the same number).

(b) Given the sum is less than 6, what is the probability of doubles?

(c) Find the probability at least one die is 5.

(d) Given that each die shows a different number, find the conditional probability that at least one die is 5.

(e) If we are told that both the product and sum of the face values are less than 7 and that at least one of the face values is a two, determine the probability the other face value is a one.

a) 6 possible doubles
36 possible rolls
$\Rightarrow P(\text{doubles}) = 6/36 = 1/6$

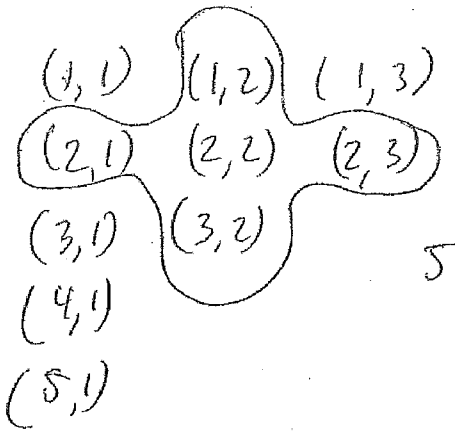
b) $A = \{$ sum less than 6 $\}$
$= \{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (3,1), (3,2), (4,1)\}$
there are 2 doubles in this set
10 elements in A $\Rightarrow P(A) = 10/36$
$P(\text{doubles} \mid A) = P(\text{doubles} \cap A)/P(A) = \frac{2/36}{10/36}$
$= 1/5$

c) 11 possible outcomes w/ at least one 5
$P(\text{at least one 5}) = 11/36$

d) 30 possible outcomes w/ different #s,
of these, 10 have at least one 5
$P(\text{at least one 5} \mid \text{diff \#}) = 10/30 = 1/3$

#13 continued

e)



5 possible w/ product and sum less than 7 and at least one 2.

2 cases within this w/ a 1.

$\Rightarrow 2/5$

#14

~~Problem 5.~~ (20 POINTS (A AND C ARE 7 POINTS; B IS 6 POINTS))
Online video streaming is achieved by attempting to match the video bandwidth to fit the bandwidth available between the video server and the viewer's computer. When more bandwidth is available, higher quality video can be achieved. However, not all movies require the same bandwidth for "acceptable" quality. (For example, action movies typically require more bandwidth than romantic comedies, for the same video quality.)

Suppose video bandwidth is divided into 3 categories: low-, medium- and high bandwidth. Further, suppose studies of viewer satisfaction show that medium-bandwidth videos are twice as likely as high-bandwidth videos to be rated as "unacceptable", but only half as likely as low-bandwidth videos to be rated "unacceptable".

(a) Suppose a company designs their system such that 50% of videos are delivered at low bandwidth, 30% at medium bandwidth, and 20% at high bandwidth.

If the probability a medium-bandwidth video is "unacceptable" is 2/7, compute the probability any video is "unacceptable".

(b) If a delivered video is rated by viewers as "unacceptable", what is the probability it was sent at high bandwidth?

(c) The company decides their system performs too poorly. They realize they can only increase the fraction of high-bandwidth videos to 0.5. What is the smallest possible probability of "unacceptable" videos they can hope to achieve given this constraint?

(You may leave all answers in fractional form.)

$U = \{$ video is unacceptable $\}$

$L = \{$ low bandwidth $\}$
$M = \{$ medium bandwidth $\}$
$H = \{$ high bandwidth $\}$

these 3 form a partition

a) theorem of total probability

$$P(U) = P(U|L)P(L) + P(U|M)P(M) + P(U|H)P(H)$$

and $P(U|L) = 2P(U|M) = 4/7$

$P(U|H) = P(U|M)/2 = 1/7$

so $P(U) = \frac{4}{7}\frac{5}{10} + \frac{2}{7}\frac{3}{10} + \frac{1}{7}\frac{2}{10} = \frac{20+6+2}{70} = \frac{28}{70} = \boxed{\frac{2}{5}}$

b) Bayes Rule: $P(H|U) = \frac{P(U|H)P(H)}{P(U)} = \frac{\frac{1}{7}\frac{2}{10}}{2/5} = \boxed{\frac{1}{14}}$

c) Set $P(H) = 1/2$, no other constraint on $P(M)$ or $P(L)$, so set $P(M) = 1/2$ and $P(L) = 0$ for best quality

Then $P(U)$, using thm total probability, is

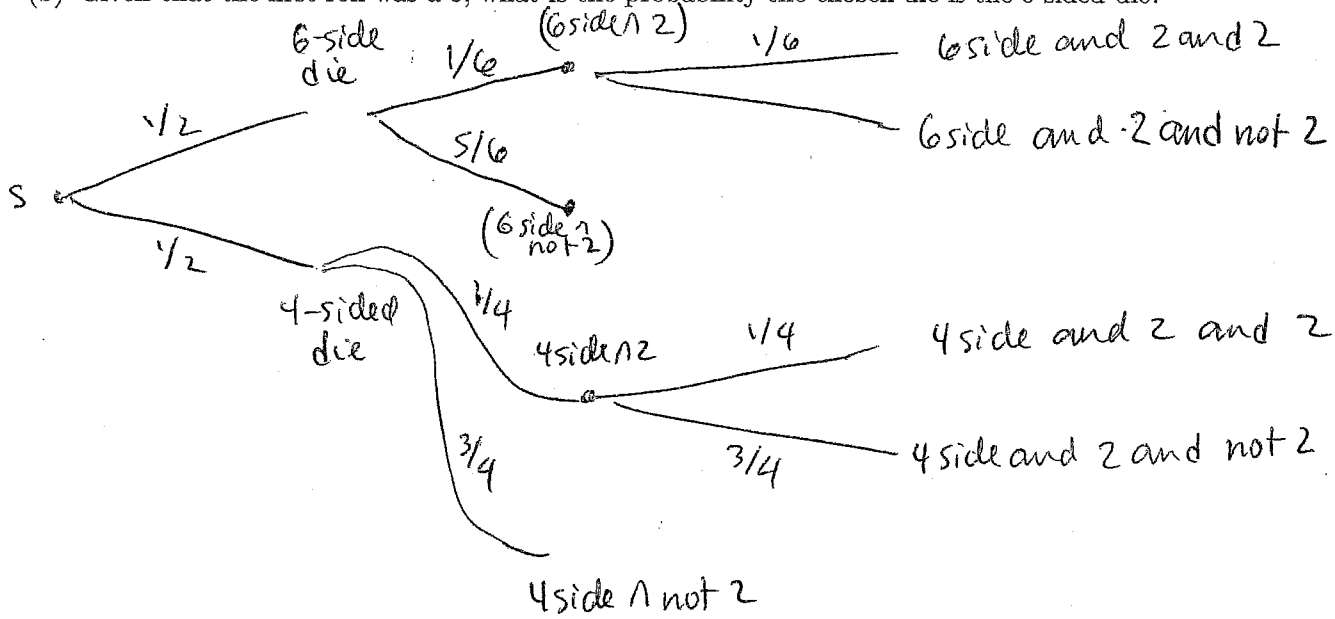
$$P(U) = \frac{4}{7}(0) + \frac{2}{7}\frac{1}{2} + \frac{1}{7}\frac{1}{2} = \boxed{\frac{3}{14}}$$

#15

~~Problem~~ 3. (15 POINTS (A IS 8 POINTS, B IS 7 POINTS))
A box contains 2 dice: a fair die with 6 sides (labelled 1,2,3,4,5,6) and a fair die with 4 sides (labelled 1,2,3,4). You select one object at random, then roll it twice.

(a) What is the probability both rolls show a 2?

(b) Given that the first roll was a 3, what is the probability the chosen die is the 6-sided die?



a) $P(\text{both sides 2})$

$= P(\text{6 sides})\,P(\text{both 2} \mid \text{6sides}) + P(\text{4 sides})\,P(\text{both 2} \mid \text{4sid})$

$= \frac{1}{2}\,\frac{1}{6}\,\frac{1}{6} + \frac{1}{2}\,\frac{1}{4}\,\frac{1}{4} = \frac{1}{2}\cdot\frac{1}{4}\left(\frac{1}{9}+\frac{1}{4}\right)$

$= \frac{13}{8\cdot 36} = \boxed{\frac{13}{288}}$

b) $P(\text{6sides} \mid \text{first 3}) = \dfrac{P(\text{first 3} \mid \text{6sides})\,P(\text{6sides})}{P(\text{first 3})}$

$P(\text{first 3}) = \frac{1}{2}P(\text{6 sides})\,P(\text{first 3} \mid \text{6 sides}) + P(\text{4 sides})\,P(\text{first 3} \mid \text{4 sides})$

$= \frac{1}{2}\,\frac{1}{6} + \frac{1}{2}\,\frac{1}{4} = \frac{1}{4}\left(\frac{1}{3}+\frac{1}{2}\right) = \frac{5}{24}$

so $P(\text{6sides} \mid \text{first 3}) = \dfrac{\frac{1}{6}\cdot\frac{1}{2}}{5/24} = \frac{1}{12}\,\frac{24}{5} = \boxed{\frac{1}{5}}$

#16

~~Problem~~ 7. (10 POINTS EACH, 30 POINTS TOTAL)
Zeros and ones are sent over a noisy communication channel, where the transmission of each bit can be considered to be independent sequential experiements. The probability that each 0 is correctly sent is 0.9, while the probability that each 1 is correctly sent is 0.85. The digit 0 is sent with probability 0.6.

(a) Find the probability that an error occurs, for each bit sent.

(b) Given that you detect a 1, what is the probability that a 1 had been sent.

(c) If the string 0010 is sent, what is the probability the string is correctly received.

Let $R_0 = \{0 \text{ received}\}$
$R_1 = \{1 \text{ received}\}$
$S_0 = \{0 \text{ sent}\}$
$S_1 = \{1 \text{ sent}\}$

we know
$P(R_0 | S_0) = 0.9 \Rightarrow P(R_1 | S_0) = 0.1$
$P(R_1 | S_1) = 0.85 \Rightarrow P(R_0 | S_1) = 0.15$
$P(S_0) = 0.6 \Rightarrow P(S_1) = 0.4$

a) $P(\text{error}) = P(R_1 \cap S_0) + P(R_0 \cap S_1)$ (these are disjoint)
$= P(R_1 | S_0) P(S_0) + P(R_0 | S_1) P(S_1)$
$= (0.1)(0.6) + (0.15)(0.4)$
$= 0.06 + 0.06 =$ ~~0.09~~ 0.12

b) $P(S_1 | R_1) = \dfrac{P(R_1 | S_1) P(S_1)}{P(R_1)}$

$P(R_1) = P(R_1 | S_1) P(S_1) + P(R_1 | S_0) P(S_0)$
so $P(S_1 | R_1) = \dfrac{(0.85)(0.4)}{(0.85)(0.4) + (0.1)(0.6)} = \dfrac{.34}{.34 + .06} = \dfrac{.34}{.40}$
$= 17/20$

c) Sequential experiments, all have to ~~happen~~ be sent correctly, independent events
$\Rightarrow (0.9)(0.9)(0.85)(0.9) = (.729)(.85)$

#17

~~Problem 4.~~ (32 POINTS)

Suppose 3 boxes contain Red, Green, and Blue marbles, denoted R, G, B, respectively.
Box 1 has 3 Red, 4 Green, and 3 Blue.
Box 2 has 8 Red, 1 Green, and 1 Blue.
Box 3 has 0 Red, 4 Green, and 1 Blue.

Suppose a box is chosen at random, and then a marble is selected from the box. A sequential experiment!

(a) If box 1 is selected, what is the probability a Green marble is drawn? $P(G|1)$

(b) What is the probability a Blue marble is drawn from Box 3? $P(B \cap 3)$

(c) What is the probability a Red marble is drawn? $P(R)$

(d) Suppose a Red marble is drawn. What is the probability it came from Box 2? $P(2|R)$

First, pick a box at random
$\Rightarrow$ each box has probability $1/3$

$P(1) = P(2) = P(3) = 1/3$

Either a table or a tree can be used here

	R	G	B	
1	$\frac{1}{3} \cdot \frac{3}{10}$	$\frac{1}{3} \cdot \frac{4}{10}$	$\frac{1}{3} \cdot \frac{3}{10}$	$1/3$
2	$\frac{1}{3} \cdot \frac{8}{10}$	$\frac{1}{3} \cdot \frac{1}{10}$	$\frac{1}{3} \cdot \frac{1}{10}$	$1/3$
3	$\frac{1}{3} \cdot 0$	$\frac{1}{3} \cdot \frac{4}{5}$	$\frac{1}{3} \cdot \frac{1}{5}$	$1/3$

If box 1 is picked,
$P(R|1) = 3/10$, $P(G|1) = \frac{4}{10}$
$P(B|1) = \frac{3}{10}$

a) $P(G|1) = \boxed{\frac{4}{10}}$

b) $P(B \cap 3) = P(B|3)\,P(3) = \frac{1}{5} \cdot \frac{1}{3} = \boxed{\frac{1}{15}}$

c) $P(R) = P(R|1)\,P(1) + P(R|2)\,P(2) + P(R|3)\,P(3)$
$= \frac{1}{3}\left(\frac{3}{10} + \frac{8}{10} + 0\right) = \boxed{\frac{11}{30}}$

d) $P(2|R) = \frac{P(R|2)\,P(2)}{P(R)} = \frac{8/10 \cdot 1/3}{11/30} = \boxed{\frac{8}{11}}$

#18

~~Problem 5.~~ (10 POINTS)

Two fair coins are each flipped once. A trustable friend tells you "I saw at least one of them was heads". Use conditional probabilities to determine the probability that both coins came up heads, given that at least one was heads.

Let $A = \{\text{at least one head}\} = \{HH, HT, TH\}$

$B = \{\text{both heads}\} = \{HH\}$

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

$P(A) = 3/4$

$P(B) = 1/4$

$$P(B|A) = \frac{1/4}{3/4} = 1/3$$

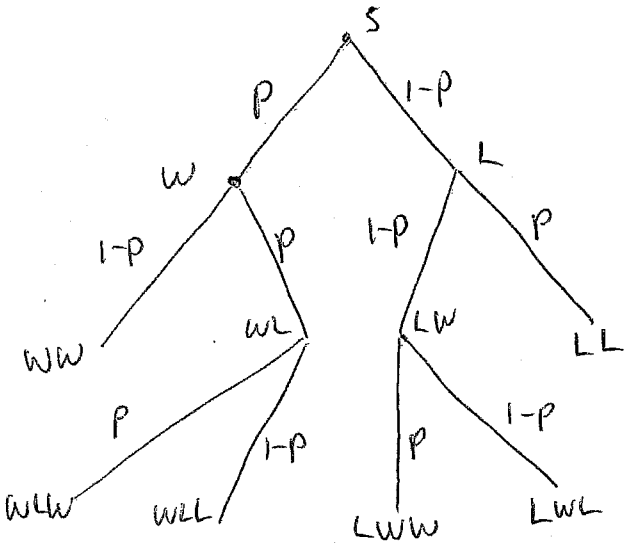
#19

~~Problem 6~~. (15 POINTS)

When two teams of equal strength play each other, the home team has a probability p of winning, where $p > 1/2$. Suppose two teams play a three-game series, where the team with home-field advantage plays first at home, then away, and then, if necessary, at home. (There is no third game if one team wins the first two games.)

What is the probability the team with home-field advantage wins the series?
(Hint: a tree diagram can be helpful here.)

BONUS 5 points: Does the home team have a greater, equal, or lesser advantage of winning a 3-game series rather than just one game? Explain why.



Consider this from the perspective of the team that starts with a home game, and draw one tree of possible outcomes.

On the first split, Prob W is P.
On the 2nd split, Prob W is 1-P
On the 3rd split Prob W is P

There are 3 outcomes which has the team winning the series: { WW, WLW, LW

$P(1^{st}$ home team wins series$)$

$= P(WW) + P(WLW) + P(LWW)$

$= p(1-p) + p^3 + (1-p)^2 p = \boxed{p[2p^2 - 3p + 2]}$

Bonus: Is $P(1^{st}$ home team wins series$) > P(1^{st}$ home team wins 1^{st} game$)$?

Is $p[2p^2 - 3p + 2] > p$?

$[2p^2 - 3p + 2] > 1$

$2p^2 - 3p + 1 > 0$

$(2p-1)(p-1) > 0$?

→ Two possibilities

ⓐ $2p-1 > 0$ and $p-1 > 0$

ⓑ $2p-1 < 0$ and $p-1 < 0$

Case ⓐ never happens for $\frac{1}{2} < p < 1$ because $p-1 < 0$

Case ⓑ only happens if $2p-1 < 0$, or $p < 1/2$.

So No, the 3-game series does NOT help the 1^{st} home team. It balances

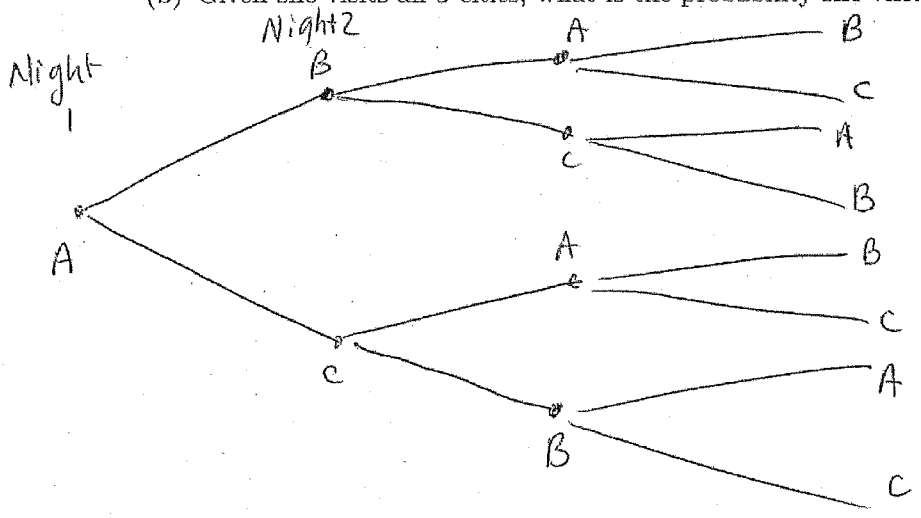
#20

~~Problem 5~~. (16 POINTS)

A salesperson travels between three cities, labeled A, B, and C, on 4 consecutive nights. Each night, she stays in only one city. On day 1 she starts at City A and stays there the first night. For the next 3 days, she chooses one of the *other* two cities at random, travels there, and stays the night. For example, if she stays in City C on night 3, on night 4 she can only stay in either City A or City B.

(a) What is the probability the salesperson visits all three cities?

(b) Given she visits all 3 cities, what is the probability she visits City A twice?



	all 3?	A twice?
ABAB	X	✓
ABAC	✓	✓
ABCA	✓	✓
ABCB	✓	X
ACAB	✓	✓
ACAC	X	✓
ACBA	✓	✓
ACBC	✓	X
	6/8	6 cases

a) $P(\text{all 3}) = \frac{6}{8} = \boxed{\frac{3}{4}}$

b) $P(\text{A twice} \mid \text{all 3}) = \frac{P(\text{A twice} \cap \text{all 3})}{P(\text{all 3})}$

$= \frac{4/8}{6/8} = \frac{4}{6} = \frac{2}{3}$

both "all 3" and "twice A" (4 cases)

21-24

The next 4 problems all refer to a discrete random variable X with the following pmf:

$$p_X(x) = \begin{cases} |x|/c & \text{for } x = -2, -1, 0, 1, 2. \\ 0 & \text{otherwise} \end{cases}$$

For these problems use the same set of answers, and clearly mark your answer next to the problem. The answers to each problem may or may not be different answers!

(a) 0

(b) 1

(c) 3

(d) 4

(e) 6

(f) None of the above

(e) **Problem 1.** (5 POINTS) Find the value of c.

(a) **Problem 2.** (5 POINTS) Find $E(X)$.

(c) **Problem 3.** (5 POINTS) Find VAR(X).

(a) **Problem 4.** (5 POINTS) Consider the random variable $Z = (X - E(X))^2$. Find $P(Z = 9)$.

(b) **Problem 5.** (5 POINTS) Consider the event $A = \{X > -1.5\}$. Find $E(X|A)$.

1) $\sum_x p_X(x) = 1 = \frac{1}{c}\left(\frac{2}{\text{---}} + 1 + 0 + 1 + 2\right)$

$= 6/c \Rightarrow \boxed{c = 6}$

2) $\boxed{E(X) = \sum_x x p_X(x) = -\frac{2 \cdot 2}{6} - \frac{1 \cdot 1}{6} + 0 + \frac{1 \cdot 1}{6} + \frac{2}{6}}$

$\boxed{= 0}$ (or, recognizing it's a symmetric pmf, $E(X) = 0$)

3) $VAR(X) = E(X^2) - E(X)^2 = E(X^2)$

since $E(X) = 0$.

$E(X^2) = \sum_x x^2 p_X(x)$

$= \left[(2)^2 \frac{2}{6} + (1)^2 \frac{1}{6}\right] 2$

$= 2\left(\frac{8}{6} + \frac{1}{6}\right) = \boxed{3}$

4)

Z	X
+4	-2
1	-1
0	0
1	1
4	2

$\boxed{P(Z = 9) =}$

cannot get 9.

P(A) =

#25

Problem 25. (TRUE/FALSE: 5 POINTS EACH, TOTAL 10 POINTS)
Label each statement T or F to the left of the problem number.

(a) $E(g(X)) = g(E(X))$ F ex: $E(X^5) \neq E(X)^5$

True (b) Let X be a random variable and let $Y = aX + b$, where a, b are constants. Then VAR(Y)=a^2VAR(X).

$$Var(Y) = E(Y^2) - E(Y)^2 =$$
$$= E\left((aX+b)^2\right) - \left(aE(X)+b\right)^2$$
$$= E\left[a^2X^2 + 2abX + b^2\right] - \left[a^2E(X)^2 + 2abE(X) + b^2\right]$$
$$= a^2E(X^2) - a^2E(X)^2 + 2abE(X) - 2abE(X)$$
$$= a^2Var(X)$$

#26 **Problem 26.** (10 POINTS)
Let X be a random variable with mean μ and variance σ^2, and let $Y = 3X + 4X^2$. Express $E(Y)$ in terms of μ and σ.

$$E(Y) = E(3X + 4X^2) = 3E(X) + 4E(X^2)$$

and

$$E(X^2) = VAR(X) + E(X)^2$$
$$= \sigma^2 + \mu^2$$

so

$$E(Y) = 3\mu + 4\sigma^2 + 4\mu^2$$

#27

Problem 10. (10 POINTS)
Suppose a basketball player shooting an uncontested 2-point shot will make the basket with probability 0.6. However, if you foul the shooter, the shot will be missed but two free throws will be awarded. Each free throw is an independent Bernoulli trial with success probability p. Based on the expected number of points the shooter will score, for what values of p may it be desirable to foul the shooter?

Problem 11. (20 POINTS)
Let X be a random variable with $S_X = \{-1, 0, 1\}$. Let $p_X(-1) = a$, $p_X(1) = b$, and $p_X(0) = 1-a-b$. This is a valid pmf if and only if $a \geq 0$, $b \geq 0$, and $a + b \leq 1$. Let $\mu = E(X)$, $\sigma^2 = \text{VAR}(X)$, and $m_2 = E(X^2)$.

(a) Find a and b so that $\mu = 1/2$ and $\sigma^2 = 3/4$.

(b) Express a and b in terms of μ and m_2. Determine and sketch the region of (μ, m_2) pairs for which there is a valid choice of (a, b).

Problem 10

Case 1: N = # points if shoots w/out foul

$E(N) = (2\text{points})(0.6) = 1.2$ points

case 2: M = #points if fouled and takes 2 free throws, each p

$P_M(0) = (1-p)^2$

$P_M(1) = 2p(1-p)$

$P_M(2) = p^2$

$E(M) = 2p(1-p) + 2p^2 = 2p$

Better to foul when $E(M) \leq E(N)$

$2p < 1.2 \Rightarrow \boxed{p < 0.6}$

#28

Problem 7. (10 POINTS)
You are manager of a ticket agency that sells concert tickets. You assume that people will call three times in an attempt to buy tickets and then give up. You want to make sure that you are able to serve at least 95% of the people who want tickets. Let p be the probability that a caller gets through to your ticket agency. What is the minimum value of p to necessary to meet your goal?

#31

Problem 8. (10 POINTS)
Suppose the probability that a car will have a flat tire while driving on Interstate I-65 is 0.0004. What is the probability that of 10000 cars driving on Interstate I-65, fewer than 3 will have a flat tire. Use the Poisson approximation.

Problem 9. (5 POINTS)
Suppose X is a continuous random variable whose probability density function is

$$f_X(x) = \begin{cases} c(4x - 2x^2) & \text{if } 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

What is the value of c?

28

Problem 7 Let N be number failures in a series of independent Bernoulli trials. Let p be the probability of success.

Want p such that no more than 5% of callers call 3 times unsuccessfully and then give up. $P(N \geq 3) < 0.05$

$P(N=n) = p(1-p)^n$, so $P(N \geq 3) = \sum_{k=3}^{\infty} p(1-p)^k = (1-p)^3 p \sum_{j=0}^{\infty} (1-p)^j$

$= (1-p)^3$

So $(1-p)^3 < 0.05$, $p > 1 - (0.05)^{1/3}$, $\boxed{p > 0.632}$

31

Problem 8 $p = 0.0004$; $n = 10000$; $\alpha = np = (4 \cdot 10^{-4})(10^4) = 4$

$P(X=k) = \frac{4^k e^{-4}}{k!}$; $P(X<3) = \sum_{k=0}^{2} \frac{\alpha^k}{k!} e^{-\alpha} = e^{-4}(1 + 4 + 8)$

$\approx \boxed{0.238}$

Problem 9 $\int_0^2 c(4x - 2x^2)\,dx = 1 = c\left(\frac{4x^2}{2} - \frac{2x^3}{3}\right)\Big|_0^2 = c(8 - 16/3)$

$c\frac{8}{3} = 1 \Rightarrow \boxed{c = 3/8}$

(g) All of the above.

#29

Problem 7. (5 POINTS)
Suppose X is the number of failures until the first success in a series of independent Bernoulli trials with probability of success 0.2. Find $P(X \geq 10)$.

X = # failures until 1st success Geometric

$p(\text{success}) = 0.2 = p$

$$P(X \geq 10) = \sum_{k=10}^{\infty} (1-p)^k p \qquad \text{let } j = k-10,\ k = j+10$$

$$= \sum_{j=0}^{\infty} (1-p)^{j+10} p$$

$$= p(1-p)^{10} \sum_{j=0}^{\infty} (1-p)^j$$

$$= \frac{p(1-p)^{10}}{1-(1-p)} = (1-p)^{10} = \boxed{(0.8)^{10}}$$

Note because X = # failures (not # trials),

$P(X=k) = (1-p)^k p$ NOT $(1-p)^{k+1} p$

30

Problem 10. (10 POINTS)

Five cars start out on a cross-country race. The probability that a car breaks down and drops out of the race is 0.2. Cars break down independently of each other.

(a) What is the probability that exactly two cars finish the race?

(b) What is the probability that at most two cars finish the race?

(c) What is the probability that at least three cars finish the race?

Binomial $(5, 0.2)$ $\quad P(N=k) = \binom{5}{k} p^k (1-p)^{5-k}$

where N = # cars that finish so $p = 0.8$ is the probability a car does not breakdown

a) $P(N=2) = \binom{5}{2}\left(\frac{4}{5}\right)^2\left(\frac{1}{5}\right)^3 = \frac{5\cdot 4}{2}\,\frac{4^2}{5^5} = \frac{10\cdot 4^2}{5^5} = 0.0512$

b) $P(N\le 2) = \sum_{k=0}^{2} \binom{5}{k} p^k (1-p)^{5-k}$

$= \binom{5}{0}(1-p)^5 + \binom{5}{1}(1-p)^4 p + \binom{5}{2}(1-p)^3 p^2$

$= \frac{1}{5^5} + \frac{5\cdot 4}{5^5} + \frac{5\cdot 4}{2}\cdot\frac{1}{5^3}\,\frac{4^2}{5^2}$

$= (1+20+160)/5^5 \approx 0.058$

c) $P(N\ge 3) = 1 - P(N\le 2) = \binom{5}{3}(1-p)^2 p^3 + \binom{5}{4}(1-p)p^4 + \binom{5}{5}p^5$

$= 10\cdot\frac{4^3}{5^5} + \frac{5\cdot 4^4}{5^5\,5} + \frac{4^5}{5^5} = \frac{640+1280+1024}{5^5}$

#28

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$c\frac{8}{3} = 1 \Rightarrow \boxed{c = 3/8}$

#32

~~Problem 4.~~ (MIX AND MATCH: 10 POINTS)
Each of the following world problems (items (1)-(5)) matches one and only one of the list of random variables (items (a)-(e)). For each world problem, identify the appropriate random variable.

e 1. Distance of a cell phone to the nearest base station

b 2. Number of active speakers in a collection of independent conversations

a 3. Fraction of defective items in a production line

d 4. Number of photons received in an optical communication system

c 5. Number of correctly transmitted bits between two erroneous bits

3 (a) Bernoulli random variable?

2 (b) Binomial random variable?

5 (c) Geometric random variable?

4 (d) Poisson random variable?

1 (e) Uniform random variable

#33

Problem 5. (TRUE OR FALSE: 5 POINTS)
A fair coin is ~~flipped~~ n times, with independent results each time. The probability there are k heads is $\binom{n}{n-k}(1/2)^n$

$$\binom{n}{n-k} = \binom{n}{k}$$

So $$\binom{n}{k} p^k (1-p)^{n-k} = \binom{n}{k} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{n-k}$$

$$= \binom{n}{k} \left(\frac{1}{2}\right)^n = \binom{n}{n-k} \left(\frac{1}{2}\right)^n$$

#26

~~Problem 4.~~ (10 POINTS)

Let X be a random variable with mean μ and variance σ^2, and let $Y = 3X + 4X^2$. Express $E(Y)$ in terms of μ and σ.

$$E(Y) = E(3X + 4X^2)$$
$$= 3E(X) + 4E(X^2)$$

We know $\sigma^2 = E(X^2) - E(X)^2 = E(X^2) - \mu^2$

so $E(X^2) = \sigma^2 + \mu^2$

Then $\boxed{E(Y) = 3\mu + 4(\sigma^2 + \mu^2)}$

#34

~~Problem 5.~~ (10 POINTS)

Consider transmitting a signal with a fixed size of 3 bits. The probability of each bit being a "1" is 3/4, independent of the other bits' values. Let N be the number of "1"'s in the signal.

(a) Find and sketch the PMF of N.

(b) What is the probability that either $N = 0$ or $N = 3$.

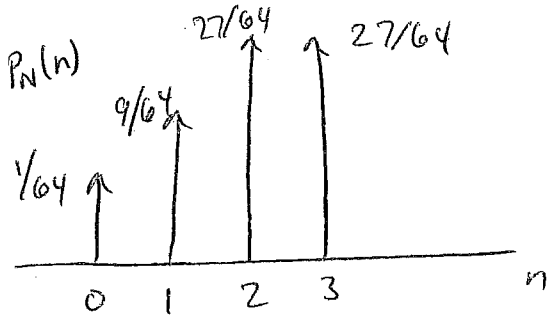
binomial

$$P_N(n) = \binom{n}{k} p^k (1-p)^{n-k}$$

$p = 3/4$, $n = 3$

a)

n	$P_N(n)$	
0	$(1/4)^3$	$= 1/64$
1	$\binom{3}{1}(1/4)^2(3/4)$	$= 9/64$
2	$\binom{3}{2}(1/4)(3/4)^2$	$= 27/64$
3	$(3/4)^3$	$= 27/64$



b) $P(N=0 \text{ or } N=3) = P(N=0) + P(N=3)$

$$= \frac{1}{64} + \frac{27}{64} = \frac{28}{64} = \boxed{\frac{7}{16}}$$

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~~Problem 7.~~ (10 POINTS)

A parking lot owner in a major city has a parking lot with 20 spaces. He has already sold monthly parking permits to 21 people, knowing that it is likely that not all cars will want to park there at the same time. In fact, the probability that an individual person will want to park their car there on any given evening is 0.98, independent of all other people.

Parking on any given night of the month is independent of every other night.

Each parking space costs \$5 a night, but if someone with a permit arrives to park and there are no spaces, the parking lot owner will refund \$10 for that night (that is, he will return the \$5 plus a pay a \$5 penalty).

Calculate the expected revenue for the parking lot owner per night.

(NOTE: You do need a complete answer, but you do not have to simplify or use a calculator. If you choose to simplify it, you may find at least one of the following expressions useful: $(0.98)^{20} = 0.67$ $(0.98)^{21} = 0.65$ $(0.02)^{20} = 10^{-34}$ $(0.02)^{21} = 2 * 10^{-36}$)

probability he will refund \$10, is the probability all 21 show up. $= (0.98)^{21}$

But he has already sold 21 permits, so he has already started with $(21)(5) = \$105$.

So his expected revenue $= 105 - 10(0.98)^{21} = 105 - 6.5$

$= 98.5$.

So, he's not very smart. He could have just sold 20 permits and made \$100!

This is a resource allocation type of problem.