# Question 3

## Fall 2019 Final

Find the explicit solution of the initial value problem

$y' = \frac{xy^2}{x^2+1}$ , $y(0) = 3$

$$\frac{dy}{dx} = \frac{xy^2}{x^2+1}$$

$$\frac{1}{y^2}\,dy = \frac{x}{x^2+1}\,dx$$

$$-y^{-1} = \int \frac{x}{x^2+1}\,dx$$

$$u = x^2+1$$

$$\frac{du}{dx} = 2x$$

$$dx = \frac{du}{2x}$$

$$= \int \frac{x}{u}\frac{du}{2x}$$

$$= \int \frac{1}{2u}\,du$$

$$\frac{-1}{y} = \frac{\ln|u|}{2} + C = \frac{\ln|x^2+1|}{2} + C$$

$$\frac{-1}{3} = \frac{\ln 1}{2} + C = 0 + C \Rightarrow C = -1/3$$

$$\frac{1}{y} = -\frac{1}{2}\ln(x^2+1) + \frac{1}{3}$$

$$y = \frac{1}{\frac{1}{3} - \frac{1}{2}\ln(x^2+1)} = \frac{6}{2 - 3\ln(x^2+1)}$$

**Question 3**

**Spring 2019 Final**

If $y = y(x)$ is the solution to

$$\frac{dy}{dx} = \frac{4xy}{2+x^2}, \qquad y(0) = 4,$$

then $y(\sqrt{3}) =$

$$\frac{1}{y}\,dy = \frac{4x}{2+x^2}\,dx$$

$u = x^2+2$

$du/dx = 2x$

$dx = \frac{du}{2x}$

$$\ln|y| = \int \frac{4x}{u}\,\frac{du}{2x}$$

$$\ln|y| = \int 2/u\,du = 2\ln(x^2+2) + C$$

$$\ln 4 = 2\ln 2 + C = \ln 4 + C \Rightarrow C = 0$$

$$\ln|y| = 2\ln 5 = \ln 25 \Rightarrow y = \pm 25$$

## Question 2

**Fall 2024 Exam 1**

Solve the following initial value problem by using the separation of variables.

$$\frac{dy}{dx} = \frac{4x^2y^2 + y^2}{x^2y^2 + x^2}, \ y(-1) = 1.$$

$$\frac{dy}{dx} = \frac{y^2(4x^2+1)}{x^2(y^2+1)}$$

$$\frac{y^2+1}{y^2} dy = \frac{4x^2+1}{x^2} dx$$

$$1 + y^{-2} dy = 4 + x^{-2} dx$$

$$y - y^{-1} = 4x - x^{-1} + C$$

$$1 - 1 = 0 = 4(-1) - (-1)^{-1} + C$$

$$0 = -4 + 1 + C = C = 3$$

$$y - \frac{1}{y} - 4x + \frac{1}{x} = 3$$

## Question 1

**Fall 2023 Exam 1**

Find the explicit solution to the initial value problem

$$\begin{cases} \frac{dy}{dx} = \frac{xy}{\sqrt{1+x^2}} \\ y(0) = 1 \end{cases}$$

$$\frac{1}{y}\,dy = \frac{x\,dx}{\sqrt{x^2+1}}$$

$$x(x^2+1)^{-1/2}$$

$$(x^2+1)^{1/2}$$

$$\ln|y| = \int \frac{x}{\sqrt{x^2+1}}\,dx$$

$$\ln|y| = (x^2+1)^{1/2} + C$$

$$\ln|1| = 0 = 1 + C \Rightarrow C = -1$$

$$\ln|y| = (x^2+1)^{1/2} - 1$$

$$y = e^{(x^2+1)^{1/2}} e^{-1}$$

## Question 1

**Spring 2024 Final**

Let $y(t)$ be the solution of the initial value problem

$$y\frac{dy}{dx} = x(y^2+1), \quad y(0)=1.$$

Find the value of $y(2)$.

$$\frac{y}{y^2+1}\,dy = x\,dx$$

$$\int \frac{y}{y^2+1}\,dy = \int x\,dx$$

$u = y^2+1$

$du/dy = 2y$

$dy = du/2y$

$$\int \frac{\cancel{y}}{u}\frac{du}{2\cancel{y}} = \frac{1}{2}\ln|y^2+1| + C$$

$$\frac{1}{2}\ln(y^2+1) = \frac{1}{2}x^2 + C$$

$$y(0)=1$$

$$\frac{1}{2}\ln 2 = \frac{1}{2}(0) + C$$

$$\ln\sqrt{2} = C$$

$$\frac{1}{2}\ln(y^2+1) = \frac{1}{2}x^2 + \ln\sqrt{2}$$

$$y^2+1 = e^{x^2+2\ln 2^{1/2}} = e^{x^2}e^{\ln 2}$$

$$y = \sqrt{e^{x^2}e^{\ln 2} - 1} \rightarrow y(2) = \sqrt{e^4 e^{\ln 2} - 1}$$

Solve the initial value problem

$$\frac{dy}{dx} = \frac{xy + y^2}{x^2}, y(1) = \frac{1}{2}$$

for $x > 0$ using the substitution $v = \frac{y}{x}$

$$\frac{dy}{dx} = \frac{xy}{x^2} + \frac{y^2}{x^2} = \frac{y}{x} + \left(\frac{y}{x}\right)^2 = v + v^2$$

$v = y/x \rightarrow y = vx$

$$\frac{dy}{dx} = \frac{dv}{dx}x + v$$

$$v + v^2 = \frac{dv}{dx}x + v$$

$$\frac{1}{v^2}dv = \frac{1}{x}dx$$

$$\int \frac{1}{v^2}dv = \int \frac{1}{x}dx$$

$$-v^{-1} = \ln|x| + C$$

$$\frac{-1}{y/x} = \frac{-x}{y} = \ln|x| + C$$

$$\frac{-1}{1/2} = \ln 1 + C$$

$$-2 = 0 + C \Rightarrow C = -2$$

$$-x/y = \ln|x| - 2$$

$$1/y = \frac{2 - \ln|x|}{x}$$

$$y = \frac{x}{2 - \ln x}$$

Find the general solution of the equation

$$y^3 \frac{dy}{dx} = (y^4 + 1)\cos x$$

$$\int \frac{y^3}{y^4+1}\,dy = \int \cos x\,dx$$

$u = y^4+1$

$\frac{du}{dy} = 4y^3$

$dy = du\,\frac{1}{4y^3}$

$$\int \frac{\cancel{y^3}}{u}\frac{1}{4\cancel{y^3}}du = \sin x + C$$

$$\int \frac{1}{4u}\,du = \frac{1}{4}\ln(y^4+1) = \sin x + C$$

$$\ln(y^4+1) = 4\sin x + C$$

$$y^4+1 = e^{4\sin x + C}$$

$$y = \left[e^{4\sin x + C} - 1\right]^{1/4}$$

$$\sqrt{y}\,dy = \frac{x+4}{\sqrt{x}}\,dx$$

$$\frac{2}{3}y^{3/2} = \int x^{1/2} + 4x^{-1/2}\,dx$$

$$\frac{2}{3}y^{3/2} = \frac{2}{3}x^{3/2} + 8x^{1/2} + C$$

$$C = \frac{2}{3}x^{3/2} + 8x^{1/2} - \frac{2}{3}y^{3/2}$$

Find a general solution of the differential equation:

$$\frac{dy}{dx} = \frac{x+4}{\sqrt{xy}}, \qquad (x>0, y>0)$$

Ⓐ $\frac{1}{2}x^{3/2} + 8x^{1/2} - \frac{2}{3}y^{2/3} = C$

Ⓑ $\frac{3}{2}x^{2/3} + 8x^{1/2} + \frac{3}{2}y^{2/3} = C$

Ⓒ $y = x + 12x^{-1} + C$

Ⓓ $y = (x^{3/2} + 12\sqrt{x})^{2/3} + C$

Ⓔ $\frac{2}{3}x^{3/2} + 8x^{1/2} - \frac{2}{3}y^{3/2} = C$