Question 4

Fall 2019 Final

A stone is dropped from rest at an initial height $h = 25$ feet above the surface of the earth. Ignoring air resistance, we assume that the acceleration of the stone is $a = -g$ where $g = 32\text{ft}/\text{s}^2$ is the gravitational acceleration. How long does the stone need to strike the ground?

$a = -32$

$v = -32t + C = -32t$

$r = -16t^2 + C = -16t^2 + 25$

$-16t^2 + 25 = 0$

$16t^2 = 25$

$t^2 = 25/16$

$t = 5/4$

Question 2

Fall 2022 Exam 2

A damped forced oscillation $x(t)$ satisfies the differential equation

$$x'' + 2x' + 5x = \cos(2t).$$

The steady periodic solution $x_{sp}(t)$ can be written in the form

$$x_{sp}(t) = C\cos(\omega t - \alpha).$$

What are the values of C, ω and α?

$x_{sp} = A\cos 2t + B\sin 2t$

$x'_{sp} = -2A\sin 2t + 2B\cos 2t$

$x''_{sp} = -4A\cos 2t - 4B\sin 2t$

$-4A\cos 2t - 4B\sin 2t - 4A\sin 2t + 4B\cos 2t + 5A\cos 2t + 5B\sin 2t = \cos 2t$

$-4A + 4B + 5A = 1$

$A + 4B = 1$

$-4B - 4A + 5B = 0$

$B = 4A$

$A + 16A = 1 \implies A = 1/17$

$B = 4/17$

$\omega = 2 \qquad C = \sqrt{1/17^2 + 16/17^2} = \frac{\sqrt{17}}{17}$

$$\alpha = \begin{cases} \arctan(B/A), & A, B > 0 \\ \pi + \arctan(B/A), & A < 0 \\ 2\pi + \arctan(B/A), & A > 0,\ B < 0 \end{cases}$$

$$\alpha = \arctan\left(\frac{4/17}{1/17}\right) = \arctan 4$$

$$x_{sp} = \frac{\sqrt{17}}{17}\cos(2t - \arctan 4)$$

Question 10

Fall 2024 Exam 1

A ball with mass $m = 1$ kg is shot upward from the ground level with the initial velocity $v(0) = v_0$. It is subject to the Earth's gravitational acceleration $g = 9.8\ m/s^2$. Air resistance is modeled by a force $k|v|$ opposite to the velocity, with $k = 2\ kg/s$.

(a) Compute the velocity $v(t)$ of the ball before it reaches its maximum height.

(b) Suppose the ball reaches its maximal height at time $t_0 = \ln 2$ seconds. Show that $v_0 = 14.7\ m/s$.

a) $F = -mg - kv = m\, dv/dt$

$$-9.8 - 2v = v'$$

$$v' + 2v = -9.8$$

$$\rho = e^{\int k\,dt} = e^{\int 2\,dt} = e^{2t}$$

$$e^{2t}v' + e^{2t}2v = e^{2t}(-9.8)$$

$$(e^{2t}v)' = -9.8e^{2t}$$

$$e^{2t}v = -4.9e^{2t} + C$$

$$v = -4.9 + Ce^{-2t}$$

$$v(0) = v_0$$

$$v(0) = -4.9 + Ce^{-2(0)} = -4.9 + C$$

$$C = 4.9 + v_0$$

$$v(t) = -4.9 + (4.9 + v_0)e^{-2t}$$

b) max height $\rightarrow$ $t = \ln 2$ sec

$$0 = -4.9 + (4.9 + v_0)e^{-2(\ln 2)}$$

$$e^{-2\ln 2} = e^{\ln 2^{-2}} = \cancel{e^{\ln \frac{1}{4}}} = 1/4$$

$$0 = -4.9 + (4.9 + v_0)\frac{1}{4}$$

$$19.6 = 4.9 + v_0$$

$$v_0 = 14.7 \text{ m/s}$$

Question 9

Spring 2024 Exam 1

A ball with mass 0.2 kg is thrown upward with initial velocity 49 m/s from the ground. There is a force due to air resistance of magnitude |v|/25 directed opposite to the velocity v (measured in m/s). How much time does it take for the ball to reach its maximum height? (Use that the gravitation acceleration $g = 9.8\ m/s^2$.)

$$F = ma - \frac{1}{25}v = m\,dv/dt$$

$$-9.8(0.2) - \frac{v}{25} = 0.2v'$$

$$\frac{1}{5}v' + \frac{1}{25}v = -9.8\,\frac{1}{5}$$

$$v' + \frac{1}{5}v = -9.8$$

$$\rho = e^{\int \frac{1}{5}dt} = e^{1/5\,t}$$

$$e^{t/5}v' + \frac{1}{5}e^{t/5}v = -9.8e^{t/5}$$

$$\left[e^{t/5}v\right]' = -9.8e^{t/5}$$

$$e^{t/5}v = \int -9.8e^{t/5}dt = -5(9.8)e^{t/5} + C$$

$$v(t) = -49 + Ce^{-t/5}$$

$$v(0) = -49 + C = 49$$

$$C = 98$$

$$v(t) = -49 + 98e^{-t/5}$$

$$0 = -49 + 98e^{-t/5}$$

$$98e^{-t/5} = 49$$

$$e^{-t/5} = 1/2$$

$$-t/5 = \ln 1/2$$

$$-t = 5\ln\tfrac{1}{2}$$

$$t = -5\ln 1/2$$

$$= -5(\ln 1 - \ln 2) = -5(0 - \ln 2)$$

$$t = 5\ln 2$$